# DRAZIN INVERSES IN JÖRGENS ALGEBRAS OF BOUNDED LINEAR OPERATORS

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Abstract

Let $X$ be a Banach space and $T$ be a bounded linear operator from $X$ to itself ($T \in B(X)$.) An operator $D \in B(X)$ is a Drazin inverse of $T$ if $TD = DT$, $D = TD^2$ and $T^k = T^{k+1}D$ for some nonnegative integer $k$. In this paper we look at the Jörgens algebra, an algebra of operators on a dual system and characterise when an operator in that algebra has a Drazin inverse that is also in the algebra. This result is then applied to bounded inner product spaces and *-algebras.

## 1. Introduction

Let $T \in B(X)$, the Banach algebra of bounded linear operators from a Banach space $X$ to itself. We shall denote the null space of $T$ as $\mathcal{N}(T)$ and the range of $T$ as $\mathcal{R}(T)$. An operator $D \in B(X)$ is a Drazin inverse of $T$ if $TD = DT$, $D = TD^2$ and $T^k = T^{k+1}D$ for some nonnegative integer $k$. The smallest such $k$ is called the index of $T$ and shall be denoted by $k = \mathrm{ind}_D(T)$.

In section 2, we summarize some known results about Drazin inverses. In sec-

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tion 3 we look at a Banach algebra called the Jörgens Algebra. This algebra is so named because K. Jörgens presented this algebra in [7] as a way to study integral operators. The algebra and its spectral theory were also studied by B. Barnes in [1]. Generalised inverses in this algebra were characterised in [11]. Examples of these algebras can be found in [7, 10].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces in normed duality. That is, suppose there is a nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ on $X \times Y$ such that for some $M > 0$,

$$|\langle x, y\rangle| \leq M \ \| x \|_X \| y \|_Y \ \text{ for all } x \in X \text{ and } y \in Y. \tag{1.1}$$

Suppose $T \in B(X)$ has an adjoint with respect to this bilinear form denoted by $T^\dagger$; i.e., $\langle Tx, y\rangle = \langle x, T^\dagger y\rangle$ for all $x \in X$ and $y \in Y$. Define the *Jörgens algebra* $J_Y(X) = \mathcal{A}$ to be

$$\mathcal{A} = \{T \in B(X) \,|\, T^\dagger \text{ exists in } B(Y)\}$$

$$\text{with norm } \ \| T \| = \max\{\| T \|_{op},\ \| T^\dagger \|_{op}\}.$$

With this defined norm, $\mathcal{A}$ is a Banach algebra [7]. $\mathcal{A}$ will denote the Jörgens algebra. Because the bilinear form is nondegenerate, an operator $T$ in $\mathcal{A}$ is uniquely determined by $T^\dagger$ and vice-versa. Note that a Jörgens algebra is a saturated algebra, or more specifically a $Y$-saturated algebra [6], [7, exercise 3.18].

In section 3 we present the main result of this paper, which is to characterise when an operator in the Jörgens Algebra has a Drazin inverse that is also in the algebra.

In section 4 we study Banach spaces that have a bounded inner product. We look at the algebra $\mathcal{B}$ of operators that have an adjoint with respect to this inner product. By defining a specific norm on this algebra, it is a Banach *-algebra. We extend the main result to this situation.

## 2. Drazin Inverses

Following the convention that for an operator $T \in B(X)$, $T^0 = I$, the identity operator, there are two useful chains of subspaces:

$$\{0\} = \mathcal{N}(T^0) \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \subseteq \cdots; \text{ and}$$

$$X = \mathcal{R}(T^0) \supseteq \mathcal{R}(T) \supseteq \mathcal{R}(T^2) \supseteq \cdots.$$

The *ascent* of an operator $T$ is the smallest nonnegative integer $k$ such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$, and will be denoted by $k = \alpha(T)$. When no such number exists, the ascent is considered infinite. The *descent* of an operator $T$ is the smallest nonnegative $k$ such that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$, and will be denoted by $k = \delta(T)$. If no such number exists, the descent is infinite. Many algebraic results can be obtained with these concepts, but a few useful ones to this paper will be mentioned.

**Theorem 2.1** ([12], Theorem 3.7). *If $T \in B(X)$ such that $\alpha(T) < \infty$ and $\delta(T) < \infty$, then they are actually equal to the same number $k$ and*

$$X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k).$$

**Theorem 2.2** ([8], Theorem 4). *Let $T \in B(X)$. Then $T$ has a Drazin inverse if and only iff $T$ has finite ascent and descent, in which case* $\mathrm{ind}_D(T) = \alpha(T) = \delta(T)$.

The following theorem and its proof can be found in [2] for the finite dimensional case and in [8] for the more general Banach space case. Again, we state it here in order to use it later.

**Theorem 2.3** ([2, 8]). *Let* $T \in B(X)$ *have Drazin inverse* $D$ *with* $\mathrm{ind}_D(T) = k$. *Then*

(1) $\mathcal{R}(D) = \mathcal{R}(T^k)$;

(2) $\mathcal{N}(D) = \mathcal{N}(T^k)$ *and*

(3) $TD = DT$ *is the projection onto* $\mathcal{R}(T^k)$ *along* $\mathcal{N}(T^k)$.

## 3. Jörgens Algebras

Before we characterise Drazin inverses in Jörgens algebras, some useful previous results from [11] will be stated. For ease of notation, for $k \in \mathbb{N}$ we shall denote $(T^k)^\dagger = (T^\dagger)^k$ by $T^{k\dagger}$.

**Lemma 1** ([11], Lemma 2). *Let* $T \in \mathcal{A}$.

(1) $\mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger)$;

(2) ${}^\perp\mathcal{R}(T^\dagger) = \mathcal{N}(T)$;

(3) ${}^\perp\mathcal{N}(T^\dagger) = \mathrm{cl}_{\mathcal{Y}}\mathcal{R}(T)$ *and*

(4) $\mathcal{N}(T)^\perp = \mathrm{cl}_{\mathcal{X}}\mathcal{R}(T^\dagger)$.

**Lemma 2** ([11], Lemma 3). *The following are true for any projection* $P \in \mathcal{A}$:

(1) $\mathcal{N}(P) = {}^\perp\mathcal{R}(P^\dagger)$;

(2) $\mathcal{R}(P) = {}^{\perp}\mathcal{N}(P^{\dagger})$;

(3) $\mathcal{R}(P^{\dagger}) = \mathcal{N}(P)^{\perp}$; *and*

(4) $\mathcal{N}(P^{\dagger}) = \mathcal{R}(P)^{\perp}$.

*Thus* $\mathcal{R}(P)$ *and* $\mathcal{N}(P)$ *are both* $\mathcal{Y}$*-closed and* $\mathcal{R}(P^{\dagger})$ *and* $\mathcal{N}(P^{\dagger})$ *are both* $\mathcal{X}$*-closed.*

Using the above facts about Drazin inverses and Jörgens algebras, a useful lemma is obtained.

**Lemma 3.** *Let* $T \in \mathcal{A}$. *If* $\delta(T) = k < \infty$, *then on these regions.* $\alpha(T^{\dagger}) \leq k$. *Similarly, if* $\delta(T^{\dagger}) = k < \infty$ *then* $\alpha(T) \leq k$. *In particular, if* $T$ *and* $T^{\dagger}$ *both have finite index, then they must have equal index.*

PROOF. Since $J_Y(X) = J_X(Y) = \mathcal{A}$, only one of the statements need to be shown. Suppose $\delta(T) = k$. Then by definition, $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$. But by Lemma 1, $\mathcal{R}(T^k)^{\perp} = \mathcal{N}(T^{k\dagger})$ and $\mathcal{R}(T^{k+1})^{\perp} = \mathcal{N}(T^{(k+1)\dagger})$. Thus $\mathcal{N}(T^{k\dagger}) = \mathcal{N}(T^{(k+1)\dagger})$ and so $\alpha(T^{\dagger}) \leq k$. ■

Now we can characterise Drazin inverses in Jörgens algebras.

**Theorem 3.1.** *Let* $T \in \mathcal{A}$ *with* $\mathrm{ind}_D(T) = k$. *Then the following are equivalent:*

(1) $T$ *has a Drazin inverse* $D \in \mathcal{A}$;

(2) $T^{\dagger}$ *has a Drazin inverse;*

(3) $\delta(T^{\dagger}) < \infty$;

(4) $\mathcal{R}(T^{(k+1)\dagger})$ *is* $\mathcal{X}$*-closed; i.e.,* $\mathcal{N}(T^k)^\perp = \mathcal{N}(T^{k+1})^\perp = \mathcal{R}(T^{(k+1)\dagger})$.

Proof. (1) $\Longrightarrow$ (2) is clear as $D^\dagger$ must be a Drazin inverse of $T^\dagger$ due to the properties of the bilinear form.

(2) $\Longrightarrow$ (1). Let $B$ be the Drazin inverse of $T^\dagger$ and $D$ the Drazin inverse of $T$. We need to show that $B = D^\dagger$. By Lemma 3, $\mathrm{ind}_D(T^\dagger) = k$. By Theorem 2.3, we also have

$$\mathcal{R}(T^\dagger B) = \mathcal{R}(B) = \mathcal{R}(T^{k\dagger}) \tag{3.1}$$

and

$$\mathcal{N}(T^\dagger B) = \mathcal{N}(B) = \mathcal{N}(T^{k\dagger}) = \mathcal{R}(T^k)^\perp = \mathcal{R}(D)^\perp \tag{3.2}$$

By Lemma 1, $\mathcal{N}(T^{k\dagger}) = \mathcal{R}(T^k)^\perp = \mathcal{R}(D)^\perp$. Using Theorem 2.1 along with (3.1), any $y \in Y$ can be uniquely expressed as $y = T^\dagger By + y_n$, where $y_n \in \mathcal{N}(T^\dagger B)$. Similarly, any $x \in X$ can be uniquely expressed as $x = TDx + x_n$, where $x_n \in \mathcal{N}(D) = \mathcal{R}(D)^\perp$. Thus

$$\begin{aligned}
\langle Dx, y\rangle &= \langle Dx, T^\dagger By\rangle + \langle Dx, y_n\rangle \\
&= \langle Dx, T^\dagger By\rangle \\
&= \langle TDx, By\rangle \\
&= \langle TDx, By\rangle + \langle x_n, By\rangle \\
&= \langle x, By\rangle.
\end{aligned}$$

Since $x$ and $y$ were arbitrary, $B = D^\dagger$ and $D \in \mathcal{A}$.

(2) $\Longrightarrow$ (3) is clear by Theorem 2.2.

(3) $\Longrightarrow$ (2). Let $\delta(T^\dagger) < \infty$. Since $\delta(T) = k$, $\alpha(T^\dagger) \le k$ by Lemma 3 and thus $\mathrm{ind}_D(T^\dagger) = k$ also. Thus $T^\dagger$ has a Drazin inverse by Theorem 2.2.

(4) $\Longrightarrow$ (3). Let $\mathcal{R}(T^{(k+1)\dagger})$ be $\mathcal{X}$-closed. By hypothesis, $\delta(T) = k = \alpha(T)$ and so by Lemma 3 $\alpha(T^\dagger) \le k$. But by Lemma 1 we have

$$\mathcal{R}(T^{(k+1)\dagger} = \mathcal{N}(T^{k+1})^\perp = \mathcal{N}(T^k)^\perp = \mathrm{cl}_{\mathcal{X}}\mathcal{R}(T^{k\dagger}). \tag{3.3}$$

Hence

$$\mathcal{R}(T^{(k+1)\dagger}) \subseteq \mathcal{R}(T^{k\dagger}) \subseteq \mathrm{cl}_{\mathcal{X}}\mathcal{R}(T^{k\dagger}) = \mathcal{R}(T^{(k+1)\dagger}) \tag{3.4}$$

and therefore $\mathcal{R}(T^{k\dagger}) = \mathrm{cl}_{\mathcal{X}}\mathcal{R}(T^{k\dagger}) = \mathcal{R}(T^{(k+1)\dagger}$. Thus $\delta(T^\dagger) \le k < \infty$.

(3) $\Longrightarrow$ (4). We have now proven that (1), (2) and (3) are equivalent, so $D \in \mathcal{A}$ and from Lemma 3, $\mathrm{ind}_D(T^\dagger) = k$ also. By Theorem 2.3, the projection $P$ onto $\mathcal{R}(T^k)$ along $\mathcal{N}(T^k)$ is $TD$ so must also be in $\mathcal{A}$. Similarly, $P^\dagger = T^\dagger D^\dagger$ is the projection onto $\mathcal{R}(T^{k\dagger})$ along $\mathcal{N}(T^{k\dagger})$. By Lemma 2, $\mathcal{R}(T^{k\dagger})$ is $\mathcal{X}$-closed. $\blacksquare$

It is indeed necessary for $\mathcal{R}(T^{(k+1)\dagger})$ to be $\mathcal{X}$-closed, and not $\mathcal{R}(T^{k\dagger})$ to be $\mathcal{X}$-closed as the following example that is discussed in [7] will illustrate.

*Example.* Consider the Jörgens algebra with $X = Y = C[0,1]$ with the standard bilinear form $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. Let $\gamma \in \mathbb{C}$ with $Re(\gamma) < 0$. Define the operator $T_\gamma \in B(C[0,1])$ by

$$T_\gamma f(x) = x^{\gamma-1}\int_0^x t^{-\gamma} f(t)\,dt, \quad x \in (0,1] \tag{3.5$a$}$$

$$T_\gamma f(0) = (1-\gamma)^{-1} f(0). \tag{3.5$b$}$$

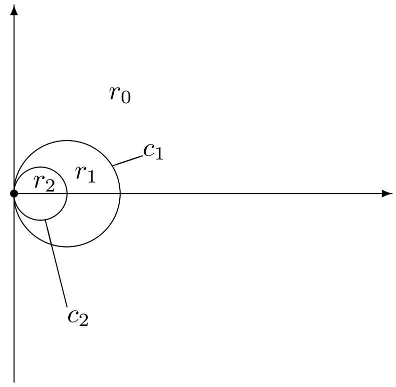


FIG. 1—Regions of the complex plane based on $a = Re(\gamma)$

It can be shown that $T_\gamma \in \mathcal{A}$ with

$$T_\gamma^\dagger f(x) = x^{-\gamma} \int_x^1 t^{\gamma-1} f(t)\, dt, \quad x \in (0,1] \tag{3.6a}$$

$$T_\gamma^\dagger f(0) = -\gamma^{-1} f(0). \tag{3.6b}$$

Consider the complex plane broken up into the following regions based on $a = Re(\gamma)$ (see figure 1)

$$\begin{aligned} c_1 &= \text{ circle with center } -\frac{1}{2a} \text{ and radius } -\frac{1}{2a} \\ c_2 &= \text{ circle with center } \frac{1}{2(1-a)} \text{ and radius } \frac{1}{2(1-a)} \\ r_0 &= \text{ region outside } c_1 \\ r_1 &= \text{ region inside } c_1 \text{ and outside } c_2 \\ r_2 &= \text{ region inside } c_2. \end{aligned}$$

We will denote the spectrum and essential spectrum of an operator $T$ by $\sigma(T)$ and $\sigma_e(T)$ and the Fredholm index will be denoted by $\iota$. It can be shown that

TABLE 1—Summary of invertibility of $\lambda - T_\gamma$ and $\lambda - T_\gamma^\dagger$

| $\lambda$ | $\lambda - T_\gamma$ | $\lambda - T_\gamma^\dagger$ |
|---|---|---|
| $r_0$ | invertible | invertible |
| $r_1$ | invertible | Fredholm, $\iota = -1$ |
| $r_2$ | Fredholm, $\iota = 1$ | Fredholm, $\iota = -1$ |
| $c_1 \backslash \{0\}$ | invertible | not Fredholm |
| $c_2 \backslash \{0\}$ | not Fredholm | Fredholm, $\iota = -1$ |
| 0 | not Fredholm | not Fredholm |

$\sigma(T_\gamma) = r_2 \cup c_2$ and $\sigma_e(T_\gamma) = c_2$. Also it can be shown that $\sigma(T_\gamma^\dagger)$ is the closed disc with boundary $c_1$ and $\sigma_e(T_\gamma^\dagger) = c_1$. In particular table 1 describes the operators $\lambda - T_\gamma$ and $\lambda - T_\gamma^\dagger$ [7, page 113].

On the regions $\lambda \in r_1 \cup c_1 \backslash \{0\}$, the operator $\lambda - T_\gamma$ is invertible and thus has a Drazin inverse with $\operatorname{ind}_D(\lambda - T_\gamma) = k = 0$. If this inverse were in $\mathcal{A}$, the operator $\lambda - T_\gamma^\dagger$ would also have to be invertible but it is not. Clearly $\mathcal{R}([\lambda - T_\gamma]^{k\dagger}) = C[0,1]$ is $\mathcal{X}$-closed and thus the hypothesis of $\mathcal{R}([\lambda - T_\gamma]^{(k+1)\dagger}) = \mathcal{R}(\lambda - T_\gamma^\dagger)$ to be $\mathcal{X}$-closed is needed.

## 4. Banach Spaces with Bounded Inner Product

As in [11], we extend Theorem 3.1 to the case where $X$ having a bounded inner product. Let $X$ be a Banach space with a bounded inner product $(\cdot, \cdot)$. For $T \in$

$B(X)$, define $T^*$ to be the adjoint of $T$ with respect to the inner product. That is,

$$(Tx, y) = (x, T^*y) \text{ for all } x, y \in X.$$

Define the algebra $\mathcal{B} = \{T \in B(X) \big| \exists T^* \in B(X)\}$. This is equivalent to the algebra of all bounded linear operators on $X$ that have bounded extensions to the Hilbert space completion of $X$ [9]. Define a norm on the elements of $\mathcal{B}$ similar to the Jörgens algebra; that is, for $T \in \mathcal{B}$,

$$\| T \| = \max\{\| T \|_{op}, \| T^* \|_{op}\}.$$

This makes $\mathcal{B}$ a Banach *-algebra and Moore-Penrose inverses in $\mathcal{B}$ were discussed in [11].

Throughout the rest of this section, $\mathcal{B}$ will denote the *-algebra above with the inner product space $X$ and $T^*$ will denote the adjoint of $T$ in this algebra. All of the results about Drazin inverses in Jörgens algebras are analogous in this setting. In particular we have the following result.

**Theorem 4.1.** *Let $T \in \mathcal{B}$ with $\mathrm{ind}_D(T) = k$. Then the following are equivalent:*

(1) *$T$ has a Drazin inverse $D \in \mathcal{B}$;*

(2) *$T^*$ has a Drazin inverse;*

(3) *$\delta(T^*) < \infty$;*

(4) *$\mathcal{R}(T^{(k+1)*})$ is $\mathcal{X}$-closed; i.e., $\mathcal{N}(T^k)^\perp = \mathcal{N}(T^{k+1})^\perp = \mathcal{R}(T^{(k+1)*})$.*

The proof of the previous lemmas and theorem are the same as in the Jörgens algebra setting as the only difference is that there is a sesquilinear rather than bilinear form.

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